Methods for solving linear systems

Reminders on norms and scalar products of vectors. The application $\| \cdot \| : \mathbb{R}^n \to \mathbb{R}$ is a norm if

$$1 - \|\underline{v}\| \geq 0, \ \forall \underline{v} \in \mathbb{R}^n \qquad \|\underline{v}\| = 0 \text{ if and only if } \underline{v} = 0;$$
$$2 - \|\alpha \underline{v}\| = |\alpha| \|\underline{v}\| \quad \forall \alpha \in \mathbb{R}, \quad \forall \underline{v} \in \mathbb{R}^n;$$
$$3 - \|\underline{v} + \underline{w}\| \leq \|\underline{v}\| + \|\underline{w}\|, \quad \forall \underline{v}, \underline{w} \in \mathbb{R}^n.$$

Examples of norms of vectors:

$$\|\underline{v}\|_2^2 = \sum_{i=1}^{n} (v_i)^2 \qquad \text{Euclidean norm}$$
$$\|\underline{v}\|_\infty = \max_{1 \leq i \leq n} |v_i| \qquad \text{max norm}$$
$$\|\underline{v}\|_1 = \sum_{i=1}^{n} |v_i| \qquad \text{1-norm}$$

Being in finite dimension, they are all equivalent, with the equivalence constants depending on the dimension n. Ex: $\|v\|_\infty \leq \|v\|_1 \leq n\|v\|_\infty$.

A scalar product is an application $(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that verifies:

$1 -$ linearity: $(\alpha\underline{v} + \beta\underline{w}, \underline{z}) = \alpha(\underline{v}, \underline{z}) + \beta(\underline{w}, \underline{z})\ \forall\alpha, \beta \in \mathbb{R},\ \forall\underline{v}, \underline{w}, \underline{z} \in \mathbb{R}^n$;
$2 - (\underline{v}, \underline{w}) = (\underline{w}, \underline{v})\ \forall\underline{v}, \underline{w} \in \mathbb{R}^n$;
$3 - (\underline{v}, \underline{v}) > 0\ \ \forall\underline{v} \neq \underline{0}$ (that is, $(\underline{v}, \underline{v}) \geq 0, (\underline{v}, \underline{v}) = 0$ iff $\underline{v} = \underline{0}$).

To a scalar product we can associate a norm defined as

$$\|\underline{v}\|^2 = (\underline{v}, \underline{v}).$$

Example: $(\underline{v}, \underline{w}) = \sum_{i=1}^{n} v_i w_i, \quad \Longrightarrow \ (\underline{v}, \underline{v}) = \sum_{i=1}^{n} v_i v_i = \|\underline{v}\|_2^2.$
(in this case we can write $(\underline{v}, \underline{w}) = \underline{v} \cdot \underline{w}$ or $\underline{v}^T \underline{w}$ for “column” vectors)

Theorem 1 (Cauchy-Schwarz inequality)

Given a scalar product $(\cdot,\cdot)_*$ and associated norm $\|\cdot\|_*$, the following inequality holds:

$$|(\underline{v},\underline{w})_*| \leq \|\underline{v}\|_*\|\underline{w}\|_* \qquad \forall v, w \in \mathbb{R}^n$$

Proof.

For $t \in \mathbb{R}$, let $t\underline{v} + \underline{w} \in \mathbb{R}^n$. Clearly, $\|t\underline{v} + \underline{w}\|_* \geq 0$. Hence:

$$\|t\underline{v} + \underline{w}\|_*^2 = t^2\|\underline{v}\|_*^2 + 2t(\underline{v},\underline{w})_* + \|\underline{w}\|_*^2 \geq 0$$

The last expression is a non-negative convex parabola in t (No real roots, or 2 coincident). Then the discriminant is non-positive

$$(\underline{v},\underline{w})_*^2 - \|\underline{v}\|_*^2\|\underline{w}\|_*^2 \leq 0$$

and the proof is concluded. □

Reminders on matrices $A \in \mathbb{R}^{n \times n}$

- A is symmetric if $A = A^T$. The eigenvalues of a symmetric matrix are real.
- A symmetric matrix A is positive definite if

$$(A\underline{x}, \underline{x})_2 > 0 \ \forall \underline{x} \in \mathbb{R}^n, \ \underline{x} \neq 0, \quad (A\underline{x}, \underline{x})_2 = 0 \text{ iff } \underline{x} = 0$$

  The eigenvalues of a positive definite matrix are positive.
- if A is non singular, $A^T A$ is symmetric and positive definite

Proof of the last statment:

- $A^T A$ is always symmetric; indeed $(A^T A)^T = A^T (A^T)^T = A^T A$.

To prove that it is also positive definite we have to show that $(A^T A\underline{x}, \underline{x})_2 > 0 \ \forall \underline{x} \in \mathbb{R}^n, \ \underline{x} \neq 0, \quad (A^T A\underline{x}, \underline{x})_2 = 0$ iff $\underline{x} = 0$. We have:

$$(A^T A\underline{x}, \underline{x})_2 = (A\underline{x}, A\underline{x})_2 = \|A\underline{x}\|_2^2 \geq 0, \text{ and } \|A\underline{x}\|_2^2 = 0 \text{ iff } A\underline{x} = \underline{0}$$

If A is non-singular (i.e., $det(A) \neq 0$), the system $A\underline{x} = \underline{0}$ has only the solution $\underline{x} = \underline{0}$, and this ends the proof.

Norms of matrices

Norms of matrices are applications from $\mathbb{R}^{m\times n}$ to $\mathbb{R}$ satisfying the same properties as for vectors. Among the various norms of matrices we will consider the norms associated to norms of vectors, called natural norms, defined as:

$$|||A||| = \sup_{\underline{v}\neq 0} \frac{\|A\underline{v}\|}{\|\underline{v}\|}$$

It can be checked that this is indeed a norm, that moreover verifies:

$$\|A\underline{v}\| \leq |||A|||\,\|\underline{v}\|, \qquad |||AB||| \leq |||A||||||B|||.$$

Examples of natural norms (of square $n \times n$ matrices)

$$\underline{v} \in \mathbb{R}; \quad \|\underline{v}\|_\infty \;\longrightarrow\; |||A|||_\infty = \max_{i=1,\cdots,n} \sum_{j=1}^{n} |a_{ij}|,$$

$$\underline{v} \in \mathbb{R}; \quad \|\underline{v}\|_1 \;\longrightarrow\; |||A|||_1 = \max_{j=1,\cdots,n} \sum_{i=1}^{n} |a_{ij}|,$$

$$\underline{v} \in \mathbb{R}; \quad \|\underline{v}\|_2 \;\longrightarrow\; |||A|||_2 = \sqrt{|\lambda_{\max}(A^T A)|}.$$

If A is symmetric, $|||A|||_\infty = |||A|||_1$, and $|||A|||_2 = |\lambda_{\text{max in abs val}}(A)|$.
Indeed, if $A = A^T$, then $\max_i \lambda_i(A^T A) = \max_i \lambda_i(A^2) = (\max_i \lambda_i(A))^2$.

The norm $|||A|||_2$ is the *spectral norm,* since it depends on the spectrum of A.